

## Four-dimensional lattice chiral gauge theories with anomalous fermion content

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**Yoshio Kikukawa**

*Institute of Physics, University of Tokyo,  
Komaba, Tokyo 153-8902, Japan  
E-mail: kikukawa@hep1.c.u-tokyo.ac.jp*

**Hiroshi Suzuki**

*Theoretical Physics Laboratory, RIKEN,  
Wako 2-1, Saitama 351-0198, Japan  
E-mail: hsuzuki@riken.jp*

**ABSTRACT:** In continuum field theory, it has been discussed that chiral gauge theories with Weyl fermions in anomalous gauge representations (anomalous gauge theories) can consistently be quantized, provided that some of gauge bosons are permitted to acquire mass. Such theories in four dimensions are inevitably non-renormalizable and must be regarded as a low-energy effective theory with a finite ultraviolet (UV) cutoff. In this paper, we present a lattice framework which enables one to study such theories in a non-perturbative level. By introducing bare mass terms of gauge bosons that impose “smoothness” on the link field, we explicitly construct a consistent fermion integration measure in a lattice formulation based on the Ginsparg-Wilson (GW) relation. This framework may be used to determine in a non-perturbative level an upper bound on the UV cutoff in low-energy effective theories with anomalous fermion content. By further introducing the Stückelberg or Wess-Zumino (WZ) scalar field, this framework provides also a lattice definition of a non-linear sigma model with the Wess-Zumino-Witten (WZW) term.

**KEYWORDS:** Renormalization Regularization and Renormalons, Lattice Gauge Field Theories, Anomalies in Field and String Theories, Gauge Symmetry.

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## 1. Introduction

In continuum field theory, it has been discussed that chiral gauge theories with Weyl fermions in anomalous gauge representations, the so-called anomalous gauge theories [1–8, 10–13], can consistently be quantized, provided that some of gauge bosons are permitted to acquire (bare) mass [14]. Such theories in four dimensions are inevitably non-renormalizable and must be regarded as a low-energy effective theory with a finite UV cutoff. On the basis of perturbation theory, it has also been argued that the UV cutoff has an upper bound given by the gauge boson mass up to a proportionally constant [14]. See also refs. [15–18].

On the other hand, following a general lattice formulation of chiral gauge theories of refs. [19, 20], it has been shown that the fermion sector of a wide class of anomalous gauge theories, that includes all four-dimensional anomalous theories, cannot consistently be defined on the lattice [21–23]. In appendix A, we present the essence of this observation in the form of a no go theorem that is quite independent of a specific lattice formulation. (Although the theorem in appendix A covers only non-abelian theories, a similar no go theorem can be established also for compact abelian theories.) See also refs. [24–28] for related studies. Although this statement is mathematically correct, the conclusion appears somewhat unnatural from a physical point of view: At this moment, we know only a low-energy (compared to, say, the Plank scale) spectrum of fermions with which the gauge anomalies are fortunately cancelled. It is quite possible, however, that a new heavy fermion

that would give rise to an additional gauge anomaly will be discovered. Then does the above statement imply that we have to suspend investigations of dynamics (based on the lattice) of such a system until other heavy fermions that complete the anomaly cancellation will be discovered? In other words, do we have to know an anomaly-free fundamental theory very precisely to study dynamics of chiral gauge theories at an (low) energy scale of our concern? This appears unnatural.

In this paper, we give an answer for the above question at least partially. We show that the fermion sector of anomalous gauge theories can consistently be formulated on the lattice, if one introduces a bare mass term of the gauge field and imposes a sufficiently strong “smoothness” condition of the link field. More definitely, we can explicitly construct a consistent Weyl fermion integration measure in the sense of refs. [19, 20] in the vacuum sector of the configuration space of link fields. Technically, such mass terms remove gauge field configurations that cause obstructions [22, 23] for a consistent fermion integration measure. Such restriction on gauge degrees of freedom was not assumed in refs. [22, 23] and thus we can evade the above conclusion on an impossibility of lattice anomalous gauge theories. See appendix A for a detailed account on these points.

The mass term of gauge bosons we introduce is not invariant under lattice gauge transformations. However, the gauge invariance is anyhow broken by fermions in an anomalous gauge representation. Introduction of a bare mass term is in fact very natural because, as is well-known, the gauge anomaly induces mass for gauge bosons through higher-order diagrams even if bare mass is set to be zero [29, 30]. Since mass terms of gauge bosons in four dimensions imply non-renormalizability, our lattice framework should be used with finite lattice spacings. (In this paper, we consider only four-dimensional spacetime.) In this way, at least for cases that all gauge bosons are massive, we have a picture in lattice gauge theory that is consistent with expectations in the continuum theory [14].

Not only for clarifying the theoretical issue elucidated above, our lattice framework could also be used for practical purposes. This framework may be used to determine in a non-perturbative level an upper bound on the UV cutoff in low-energy effective theories with anomalous fermion content. This possibility in lattice gauge theory was first suggested in ref. [31]. If we further introduce the Stückelberg or WZ scalar field, this framework provides also a lattice definition of a four-dimensional non-linear sigma model with the WZW term. Thanks to a lattice Dirac operator that satisfies the GW relation [32], such as the overlap Dirac operator [33, 34], the WZW term has expected topological properties [22].

Throughout this paper, the spacetime dimension is set to be four. Greek letters,  $\mu, \nu, \dots$ , run from 0 to 3. We consider a four-dimensional square lattice

$$\{x \in a\mathbb{Z}^4 \mid 0 \leq x_\mu < L\}, \quad (1.1)$$

where  $a$  denotes the lattice spacing. A unit vector in, say, the  $\mu$ -direction is denoted by  $\hat{\mu}$ . For definiteness, the gauge group  $G$  is taken to be  $SU(N)$ , but inclusion of  $U(1)$  factors and other  $SU(N')$  factors is straightforward. The standard link variables are denoted by  $U(x, \mu) \in G$ . We assume that a Weyl fermion belongs to a unitary (anomalous and generally irreducible) representation  $R$  of  $G$ .

## 2. Lattice formulation

### 2.1 General framework

The expectation value of an operator  $\mathcal{O}$  in our lattice framework is defined by

$$\langle \mathcal{O} \rangle = \frac{1}{\mathcal{Z}} \int \prod_x \prod_\mu dU(x, \mu) e^{-S_G[U] - S_{\text{mass}}[U]} \langle \mathcal{O} \rangle_{\text{F}} [U], \quad (2.1)$$

where  $dU(x, \mu)$  denotes the standard Haar measure and

$$\mathcal{Z} = \int \prod_x \prod_\mu dU(x, \mu) e^{-S_G[U] - S_{\text{mass}}[U]} \langle 1 \rangle_{\text{F}} [U] \quad (2.2)$$

is the full partition function. The functional integration with respect to a Weyl fermion is given by

$$\langle \mathcal{O} \rangle_{\text{F}} [U] = \int D[\psi] D[\bar{\psi}] \mathcal{O} e^{-S_{\text{F}}[\psi, \bar{\psi}, U]} \quad (2.3)$$

and, as usual, fermion fields in  $\mathcal{O}$  are Wick-contracted by the fermion propagator that can be read off from the action  $S_{\text{F}}$ . What is non-trivial is a construction of the Weyl determinant  $\langle 1 \rangle_{\text{F}} [U]$  or, equivalently, a definition of the fermion integration measure  $D[\psi] D[\bar{\psi}]$ . The basic idea of our framework is simple but a construction is somewhat complex. Thus we explain a definition of actions part by part in following subsections. A detailed account on the fermion integration measure (that is a crucial part of our framework) will be given in the next section.

### 2.2 Modified plaquette action $S_G$

We start with a definition of the gauge action  $S_G$ . It is given by

$$S_G[U] = \frac{1}{g_0^2} \sum_x \sum_{\mu\nu} \mathcal{L}_{\mu\nu}(x), \quad (2.4)$$

where  $g_0$  denotes the bare gauge coupling constant and the functions  $\mathcal{L}_{\mu\nu}(x)$  are defined by

$$\mathcal{L}_{\mu\nu}(x) = \begin{cases} \frac{\text{Re tr} \{1 - \mathcal{P}_{\mu\nu}(x)\}}{1 - \text{Re tr} \{1 - \mathcal{P}_{\mu\nu}(x)\} / f_R(\epsilon)} & \text{if } \text{Re tr} \{1 - \mathcal{P}_{\mu\nu}(x)\} < f_R(\epsilon), \\ +\infty & \text{otherwise,} \end{cases} \quad (2.5)$$

from the plaquette variables

$$\mathcal{P}_{\mu\nu}(x) = U(x, \mu) U(x + a\hat{\mu}, \nu) U(x + a\hat{\nu}, \mu)^{-1} U(x, \nu)^{-1}. \quad (2.6)$$

In eq. (2.5),  $\epsilon$  is a constant being independent of gauge-field configurations. The action  $S_G$  is a modified plaquette action [19] which dynamically imposes the restriction  $\text{Re tr} \{1 - \mathcal{P}_{\mu\nu}(x)\} < f_R(\epsilon)$  for all  $x, \mu$  and  $\nu$  on gauge-field configurations. It can be shown that, with an appropriate choice of the function  $f_R(\epsilon)$  that depends also on a gauge group

representation  $R$  of the Weyl fermion, the restriction implies the so-called admissibility condition [35–37]

$$\|1 - R[\mathcal{P}_{\mu\nu}(x)]\| < \epsilon \quad \text{for all } x, \mu, \nu. \quad (2.7)$$

In this expression,  $\|A\|$  denotes the matrix norm, i.e., the square root of the maximal eigenvalue of  $A^\dagger A$ , and  $R$  denotes the gauge group representation of the Weyl fermion.<sup>1</sup> For instance, we can take  $f_R(\epsilon) = \epsilon^2/2$  for the fundamental representation of  $SU(N)$  and  $f_R(\epsilon) = \epsilon^2/8$  for the adjoint representation of  $SU(N)$ . Note that the admissibility is a gauge invariant condition. That is, it is invariant under the gauge transformation

$$U(x, \mu) \rightarrow U^\Lambda(x, \mu) = \Lambda(x)U(x, \mu)\Lambda(x + a\hat{\mu})^{-1}, \quad (2.8)$$

where  $\Lambda(x) \in G$ .

The modified plaquette action (2.4) defines the Boltzmann weight  $e^{-S_G}$  that is a product of local  $C^\infty$  functions of link variables. It differs from the standard plaquette action substantially only for field configurations in which the field strength is of the order of the UV cutoff,  $O(1/a^2)$ . Thus the modification can be regarded as a part of allowable lattice artifacts. It is quite conceivable that the modified action belongs to the same universality class as conventional gauge actions in the weak coupling region.

An implication of the admissibility (2.7) is two-fold. First, it ensures that the overlap-Dirac operator [33, 34], that we will adopt below, is well-defined and local [36, 37] if  $\epsilon$  is less than  $1/[6(2 + \sqrt{2})]$ . Second, the admissibility divides the space of lattice gauge-field configurations into “topological sectors” [35]. In fact, these two facts are closely related to each other through the lattice index theorem [38–41]. In the most part of this paper, we will consider the vacuum sector in the space of gauge-field configurations, that is, one of topological sectors that contains the trivial vacuum  $U(x, \mu) \equiv 1$ . A possible generalization to non-trivial topological sectors will briefly be mentioned at the very end of this paper.

The space of admissible gauge fields specified by eq. (2.7) generally possesses a non-trivial topological structure. At this moment, a parametrization of the space is known only for  $G = U(1)$  [19]. This fact is one of main obstacles for a generalization of a construction of abelian lattice chiral gauge theories [19, 42–44] to non-abelian theories. In the present context, we can avoid this difficulty by further restricting the space of gauge-field configurations within a ball enclosing the trivial vacuum  $U(x, \mu) \equiv 1$ . This is an important role of the mass term of gauge fields that we will explain next.

### 2.3 Mass term $S_{\text{mass}}$

We introduce a mass term of gauge fields of the form

$$S_{\text{mass}}[U] = \frac{2m_0^2 a^2}{g_0^2} \sum_x \sum_\mu \mathcal{M}_\mu(x), \quad (2.9)$$

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<sup>1</sup>For subsequent discussions, it is useful to note the relation  $\|1 - U\| = \sqrt{\max_i 2(1 - \cos \theta_i)}$  for a unitary matrix  $U$  with eigenvalues  $e^{i\theta_i}$ .

where

$$\mathcal{M}_\mu(x) = \begin{cases} \frac{\text{Re tr} \{1 - U(x, \mu)\}}{1 - \text{Re tr} \{1 - U(x, \mu)\} / f_R(\delta)} & \text{if } \text{Re tr} \{1 - U(x, \mu)\} < f_R(\delta), \\ +\infty & \text{otherwise.} \end{cases} \quad (2.10)$$

In eq. (2.10),  $\delta$  is a constant being independent of gauge-field configurations. As the modified plaquette action (2.4), this mass term  $S_{\text{mass}}$  dynamically imposes the condition  $\text{Re tr} \{1 - U(x, \mu)\} < f_R(\delta)$  and, as before, this implies the condition

$$\|1 - R[U(x, \mu)]\| < \delta \quad \text{for all } x \text{ and } \mu, \quad (2.11)$$

which we will refer to as the “smooth” condition. Note that this condition is *not* gauge invariant, because it is a condition on link variables that transform as eq. (2.8). This is not so surprising, because mass terms of gauge bosons are anyhow not gauge invariant. The associated Boltzmann weight  $e^{-S_{\text{mass}}}$  is a product of local  $C^\infty$ -class functions of link variables. We choose the constant  $\delta$  such that

$$\delta \leq \sqrt{2} \sqrt{1 - \cos \{\pi / (N - 1)\}} \quad \text{for } G = \text{SU}(N), \quad (2.12)$$

and

$$4\delta + 6\delta^2 + 4\delta^3 + \delta^4 \leq \epsilon. \quad (2.13)$$

With the mass term (2.9), the above system is non-renormalizable in the weak coupling expansion around the trivial vacuum  $U(x, \mu) \equiv 1$ . Although this action differs from the would-be standard mass term  $\text{Re tr} \{1 - U(x, \mu)\}$  by the factor  $[1 - \text{Re tr} \{1 - U(x, \mu)\} / f_R(\delta)]^{-1}$ , the modification becomes effective only when the gauge potential becomes the cutoff order  $O(1/a)$ . Thus it can be regarded as a part of allowable lattice artifacts.

Now, let us explain the meaning of the inequality (2.12). Under this, the space of smooth gauge-field configurations, specified by eq. (2.11), is contractible. The space is thus topologically trivial and looks like a ball with the “radius”  $\delta$ . This fact can be shown by defining a one-parameter family of gauge-field configurations

$$U_t(x, \mu) = [U(x, \mu)]^t \in G, \quad 0 \leq t \leq 1. \quad (2.14)$$

The power  $[U]^t$  of a unitary matrix  $U \in G$  can be defined as follows. Suppose that a unitary matrix  $U \in \text{SU}(N)$  satisfies  $\|1 - U\| < \delta$  and the constant  $\delta$  fulfills the inequality (2.12). Then, such a matrix can be represented as

$$U = V \text{diag}(e^{i\theta_1}, e^{i\theta_2}, \dots, e^{i\theta_N}) V^{-1}, \quad V \in G, \quad (2.15)$$

where all angles  $\theta_i$  are in the *open* interval  $-\pi < \theta_i < +\pi$ . The power  $[U]^t$  with  $0 \leq t \leq 1$  can then be defined by

$$[U]^t = V \text{diag}(e^{it\theta_1}, e^{it\theta_2}, \dots, e^{it\theta_N}) V^{-1} \in G. \quad (2.16)$$

Thus under the smooth condition (2.11) with  $\delta$  that fulfills the inequality (2.12), we can unambiguously define the one-parameter family (2.14). The one-parameter family (2.14)

continuously connects any smooth gauge-field configuration  $U$  to the trivial configuration,  $U \equiv 1$ . Thus the space of smooth gauge field configurations is contractible.

The meaning of the another inequality (2.13) is as follows. Under eq. (2.13), any configuration which satisfies the smooth condition (2.11) is admissible, that is, it satisfies eq. (2.7). This can be seen by applying the Schwartz inequality for the matrix norm to eq. (2.7). Moreover, from the above construction, we see that if  $U(x, \mu)$  is admissible then the one-parameter family (2.14) is also admissible.

These explain the origin of inequalities (2.12) and (2.13). Under eq. (2.12), the space specified by the condition (2.11) is a contractible ball. If  $\delta$  fulfills the inequality (2.13), the ball is moreover contained in the space of admissible gauge-field configurations. In this way, we restrict possible gauge-field configurations into a topologically trivial space. One would be afraid of that such restriction is too strong, i.e., the condition (2.11) excises also gauge-field configurations which become physically important in the continuum limit. If the link variables can be expanded by the gauge potentials as  $U(x, \mu) \simeq 1 + aA_\mu(x)$  in the continuum limit, as we are assuming, the condition becomes  $|A_\mu(x)| \lesssim \delta/a$  in the continuum limit. It is then clear that nothing important is lost in the  $a \rightarrow 0$  limit.

Recall that the restriction (2.11) is not invariant under lattice gauge transformations. We need such a non gauge invariant restriction to avoid the no go theorem in appendix A. This restriction moreover allows one to construct a consistent fermion integration measure, while evading difficulty of finding a precise parametrization of the space of admissible configurations.

On the other hand, for the above simple trick to work, it is clear that we have to introduce mass terms for *all* gauge bosons. In the present framework, one cannot keep some of gauge bosons, those associated with an (anomaly-free) unbroken subgroup  $H$ , massless. This limits a range of applicability of the present lattice framework.

## 2.4 Fermion action $S_F$

We now turn to the fermion action defined by

$$S_F[\psi, \bar{\psi}, U] = a^4 \sum_x \bar{\psi}(x) D\psi(x). \quad (2.17)$$

The lattice Dirac operator  $D$  is assumed to satisfy the GW relation [32]

$$\gamma_5 D + D\gamma_5 = aD\gamma_5 D, \quad (2.18)$$

that implies an exact chiral symmetry on the lattice [41]. For definiteness, we assume use of the overlap-Dirac operator [33, 34] in what follows.

We first introduce the modified chirality matrix [45, 46]  $\hat{\gamma}_5 = \gamma_5(1 - aD)$ . This operator satisfies

$$(\hat{\gamma}_5)^\dagger = \hat{\gamma}_5, \quad (\hat{\gamma}_5)^2 = 1, \quad D\hat{\gamma}_5 = -\gamma_5 D, \quad (2.19)$$

where the last two relations follow from the GW relation (2.18). One then defines projection operators

$$\hat{P}_\pm = \frac{1}{2}(1 \pm \hat{\gamma}_5), \quad P_\pm = \frac{1}{2}(1 \pm \gamma_5). \quad (2.20)$$

Note that the hatted projection operators  $\hat{P}_\pm$  depend on a gauge-field configuration through the Dirac operator  $D$ . It is then assumed that the fermion variables are subject to the following constraints specifying the (left-handed) chirality

$$\hat{P}_-\psi(x) = \psi(x), \quad \bar{\psi}(x)P_+ = \bar{\psi}(x). \quad (2.21)$$

This construction of a lattice action of a Weyl fermion is equivalent to the domain-wall formulation [47] and to the overlap formulation [48–55] of lattice Weyl fermions.

### 3. Construction of the fermion integration measure

To define the integration measure  $D[\psi]D[\bar{\psi}]$  for a Weyl fermion in eq. (2.3), one introduces orthonormal bases in the constrained spaces (2.21):

$$\hat{P}_-v_j(x) = v_j(x), \quad (v_k, v_j) = \delta_{kj}, \quad (3.1)$$

$$\bar{v}_k(x)P_+ = \bar{v}_k(x), \quad (\bar{v}_j^\dagger, \bar{v}_k^\dagger) = \delta_{kj} \quad (3.2)$$

and expand field variables as

$$\psi(x) = \sum_j v_j(x)c_j, \quad \bar{\psi}(x) = \sum_k \bar{c}_k \bar{v}_k(x). \quad (3.3)$$

The fermion integration measure is then defined by

$$D[\psi]D[\bar{\psi}] = \prod_j dc_j \prod_k d\bar{c}_k \quad (3.4)$$

in terms of the Grassmann expansion coefficients.

In eq. (3.1), basis vectors  $\{v_j\}$  depend on the gauge-field configuration through the projection operator  $\hat{P}_-$ . However, eq. (3.1) does not fix basis vectors uniquely (any unitary transformation of  $\{v_j\}$  with respect to the index  $j$  leaves the constraint invariant). This arbitrariness of basis vectors results in phase ambiguity of the fermion integration measure that may depend on the gauge-field configuration.<sup>2</sup> One has to fix this ambiguity so that the locality and smoothness<sup>3</sup> hold [19, 20].

To study this problem, it is convenient to introduce the *measure term* [19, 20]

$$\mathfrak{L}_\eta = a^4 \sum_x \eta_\mu^a(x) j_\mu^a(x), \quad (3.5)$$

where  $\eta^a(x)$  denotes a variation vector of link variables

$$\delta_\eta U(x, \mu) = a\eta_\mu(x)U(x, \mu), \quad \eta_\mu(x) = \eta_\mu^a(x)T^a, \quad (3.6)$$

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<sup>2</sup>The basis vectors  $\{\bar{v}_k\}$  in eq. (3.2) can be taken to be independent of a gauge-field configuration and there is no ambiguity associated with their choice.

<sup>3</sup>Here, the smoothness means that any expectation value in the fermion sector (2.3) is a single-valued  $C^\infty$ -class function of link variables. For anomaly-free chiral gauge theories, the phase moreover must be consistent with the gauge invariance.



and  $T^a$  are anti-hermitian generators of  $G$ . The current  $j_\mu^a(x)$ , that is a function of a gauge-field configuration, is referred to as the measure current.

It can be shown [19, 20] that if a given measure term satisfies several prerequisites, one can reconstruct the fermion integration measure that is consistent with the locality and smoothness. Generally, those prerequisites are quite non-trivial to be fulfilled if the space of gauge-field configurations possesses a non-trivial topological structure (such as non-contractible loops). In our present system, gauge-field configurations are restricted by the smooth condition (2.11) and the space of allowed gauge-field configurations is topologically trivial. For such a topologically trivial configuration space, the above prerequisites are reduced to the locality and the local integrability. The locality here means that the measure current  $j_\mu^a(x)$  is a local expression of link variables. The local integrability is

$$\delta_\eta \mathcal{L}_\zeta - \delta_\zeta \mathcal{L}_\eta + a \mathcal{L}_{[\eta, \zeta]} = i \text{Tr} \left\{ \hat{P}_- [\delta_\eta \hat{P}_-, \delta_\zeta \hat{P}_-] \right\}, \quad (3.7)$$

where variations  $\eta$  and  $\zeta$  are assumed to be independent of link variables.

In our present system, in fact, it is easy to construct a measure term that fulfills the locality and the local integrability (3.7). Denoting the projection operator associated with the one-parameter family (2.14) as

$$P_t = \hat{P}_- \Big|_{U=U_t}, \quad (3.8)$$

we can adopt the following measure term

$$\mathcal{L}_\eta = i \int_0^1 dt \text{Tr} \{ P_t [\partial_t P_t, \delta_\eta P_t] \}. \quad (3.9)$$

It is clear that, from the locality of the overlap Dirac operator [36, 37], the measure current associated with the above measure term is a local expression of link variables. Thus the locality is ensured. One can also confirm that this measure term satisfies the local integrability (3.7). These allows us to construct basis vectors  $\{v_j\}$  that are consistent with the locality and smoothness [19, 20].<sup>4</sup>

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<sup>4</sup>The construction proceeds as follows: One introduces a unitary operators  $Q_t$  by the differential equation,

$$\partial_t Q_t = [\partial_t P_t, P_t] Q_t, \quad Q_0 = 1. \quad (3.10)$$

One also computes the Wilson line associated with the measure term by

$$W = \exp \left\{ i \int_0^1 dt \mathcal{L}_\eta \right\}, \quad a \eta_\mu(x) = \partial_t U_t(x, \mu) U_t(x, \mu)^{-1}. \quad (3.11)$$

For our measure term (3.9), we have  $\mathcal{L}_\eta = 0$  for  $a \eta_\mu = \partial_t U_t U_t^{-1}$  and thus  $W = 1$ . Finally, one makes a certain choice of basis vectors  $\{w_j\}$  for the vacuum  $U \equiv 1$ . From these, basis vectors for  $U$  is given by

$$v_j = Q_1 w_j \quad (3.12)$$

and the Weyl determinant [20]

$$\langle 1 \rangle_F [U] \langle 1 \rangle_F [1]^* = \det \left\{ 1 - P_+ + P_+ D Q_1 D_0^\dagger \right\}, \quad (3.13)$$

where  $D_0$  is the Dirac operator for  $U \equiv 1$ .

In particular, a variation of the fermion effective action is given by

$$\delta_\eta \ln \langle 1 \rangle_F [U] = \text{Tr} \left\{ \delta_\eta D \hat{P}_- D^{-1} P_+ \right\} - i \mathfrak{L}_\eta \quad (3.14)$$

and its integration along the path (2.14) gives the effective action (see also ref. [56])

$$\begin{aligned} \ln \langle 1 \rangle_F [U] - \ln \langle 1 \rangle_F [1] &= \int_0^1 dt \left( \text{Tr} \left\{ \partial_t D \hat{P}_- D^{-1} P_+ \right\} - i \mathfrak{L}_\eta \right) \Big|_{U=U_t} \\ &= \int_0^1 dt \text{Tr} \left\{ \partial_t D \hat{P}_- D^{-1} P_+ \right\} \Big|_{U=U_t}, \end{aligned} \quad (3.15)$$

where the variation is given by  $a\eta_\mu(x) = \partial_t U_t(x, \mu) U_t(x, \mu)^{-1}$  and we have noted  $\mathfrak{L}_\eta = 0$  for this variation  $\eta_\mu$ . This completes a construction of the fermion sector.

Note that our construction works even for a single four-dimensional Weyl fermion in the fundamental representation of  $SU(2)$ , that suffers from the  $SU(2)$  anomaly [57, 58]. It has been shown [59, 60] that the lattice formulation in refs. [19, 20] neatly reproduces the  $SU(2)$  anomaly, as a non-integrability along a non-contractible loop in the space of admissible gauge-field configurations. As we emphasized, such a topologically non-trivial structure is removed from our space of allowed gauge-field configurations and this is the reason why we can construct a consistent fermion integration measure for a Weyl fermion in any gauge representation. One may wonder, then, whether our construction is potentially inconsistent if it is applicable even to a single  $SU(2)$  Weyl fermion.

The key is again the presence of the mass term (2.9). It is not invariant under any non-trivial gauge transformation. As pointed out in ref. [14], if the action is not gauge invariant, the argument [57] that shows inconsistency of a gauge theory containing a single  $SU(2)$  Weyl fermion does not apply. One may have well-defined expectation values if the action contains, say, mass terms of gauge bosons. In this way, we again have a consistent picture.

#### 4. Introducing the Stückelberg or WZ scalar

For some purposes, it is useful to introduce the degrees of freedom of a  $G$ -valued Stückelberg or WZ scalar  $g(x)$  into our system. We multiply eq. (2.1) by unity,  $1 = \int \prod_x dg(x)$ , where  $dg(x)$  is the Haar measure, to yield

$$\langle \mathcal{O} \rangle = \frac{1}{\mathcal{Z}} \int \prod_x dg(x) \int \prod_x \prod_\mu dU(x, \mu) e^{-S_G[U] - S_{\text{mass}}[U^g]} \langle \mathcal{O} \rangle_F [U^g], \quad (4.1)$$

where we have made change of variables from  $U(x, \mu)$  to

$$U^g(x, \mu) = g(x) U(x, \mu) g(x + a\hat{\mu})^{-1}, \quad (4.2)$$

and used the gauge invariance of the gauge action  $S_G$  and of the measure.

In this new picture, the integrand of the functional integration is gauge invariant. That is,  $e^{-S_G[U] - S_{\text{mass}}[U^g]} \langle \mathcal{O} \rangle_F [U^g]$  is invariant under the gauge transformations

$$U(x, \mu) \rightarrow \Lambda(x) U(x, \mu) \Lambda(x + a\hat{\mu})^{-1}, \quad g(x) \rightarrow g(x) \Lambda(x)^{-1}, \quad (4.3)$$

because the combination  $U^g$  is trivially invariant under these transformations. This realization of gauge invariance may be regarded as anomaly cancellation between a Weyl fermion and the WZ scalar  $g$ . We may define the WZW term [61, 62] in lattice gauge theory by

$$e^{-i\Gamma_{\text{WZW}}[g^{-1}, U]} = \frac{\langle 1 \rangle_{\text{F}}[U^g]}{\langle 1 \rangle_{\text{F}}[U]}, \quad (4.4)$$

where  $g(x)$  is a  $G$ -valued scalar field (the WZ scalar).<sup>5</sup> Then the Weyl determinant  $\langle 1 \rangle_{\text{F}}[U^g]$  in eq. (4.1) can be expressed as  $e^{-i\Gamma_{\text{WZW}}[g^{-1}, U]} \langle 1 \rangle_{\text{F}}[U]$ . The Weyl determinant  $\langle 1 \rangle_{\text{F}}[U]$  is not gauge invariant, but a gauge variation of the WZW term compensates this breaking of the gauge symmetry. This anomaly cancellation can also be regarded as a simplest variant of the Green-Schwarz anomaly cancellation mechanism [63–65].<sup>6</sup>

As emphasized in ref. [14], however, the above gauge invariance in the new picture has no immediate consequence. In fact, as eq. (4.1) shows, the expectation value of any operator (such as Wilson lines) is the same as that in the old picture (2.1). If one wishes, the unitary gauge can be taken in which

$$g(x) \equiv 1 \quad (4.5)$$

by using the gauge invariance in the new picture. Then the system reduces to the old one. Thus, which picture (non gauge invariant or gauge invariant) we take is just a matter of description.

Incidentally, on the lattice, we can always take the unitary gauge (4.5) and this justifies in a non-perturbative level the treatment in ref. [14] that assumes the absence of global obstructions to set  $g(x) \equiv 1$ .

In the new picture (4.1), the mass term becomes a gauge invariant kinetic term of the WZ scalar

$$S_{\text{mass}}[U^g] = K \sum_x \sum_{\mu} \mathcal{M}_{\mu}(x), \quad K \equiv \frac{2m_0^2}{g_0^2} a^2 \quad (4.6)$$

where

$$\mathcal{M}_{\mu}(x) = \begin{cases} \frac{\text{Re tr} \{1 - g(x)U(x, \mu)g(x + a\hat{\mu})^{-1}\}}{1 - \text{Re tr} \{1 - g(x)U(x, \mu)g(x + a\hat{\mu})^{-1}\} / f_R(\delta)} & \text{if } \text{Re tr} \{1 - g(x)U(x, \mu)g(x + a\hat{\mu})^{-1}\} < f_R(\delta), \\ +\infty & \text{otherwise.} \end{cases} \quad (4.7)$$

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<sup>5</sup>Our choice of the measure term (3.9) and, as a result, a definition of the lattice WZW term (4.4) are completely identical to those of ref. [22]. We can thus repeat arguments of ref. [22] for the lattice WZW term (4.4). In particular, we can see that the WZW term is a *local* functional of  $g$  and  $U$  and possesses topological properties common to the continuum [62] even with finite lattice spacings. It can be shown that it also has a correct classical continuum limit.

<sup>6</sup>Note that, however, in general Green-Schwarz mechanism in which an anti-symmetric tensor field  $B$  (instead of a scalar field) cancels the anomaly, it is impossible to take a “unitary gauge” that completely eliminates  $B$ , because  $B$  is transformed into a Chern-Simons form by the gauge transformation, instead into a function.

Since the Weyl determinant produces the WZW term, bosonic sector of our system is just the gauged non-linear sigma model with the WZW term.<sup>7</sup> Here we assume that the parameter  $K$  is sufficiently large so that the system is in the Higgs phase and the expansion of link variables around  $U \equiv 1$  is justified in the weak coupling limit,  $g_0 \rightarrow 0$ . There is a possibility that the sigma model is always in the Higgs phase for all values of  $K$  when  $\delta$  is sufficiently small. This question is highly dynamical, especially with the presence of Weyl fermions, and is beyond the scope of this paper.

## 5. Topological sectors

In this final section, we make a brief comment on a possible generalization of our construction to topologically non-trivial sectors. As noted, the space of gauge-field configurations is divided into topological sectors under the admissibility condition (2.7). The fermion integration measure has to be defined sector by sector [19, 20]. A natural generalization of the mass term is

$$S_{\text{mass}}[U] = \frac{2m_0^2 a^2}{g_0^2} \sum_x \sum_\mu \mathcal{M}_\mu(x), \quad (5.1)$$

where

$$\mathcal{M}_\mu(x) = \begin{cases} \frac{\text{Re tr} \{1 - U_0(x, \mu)^{-1} U(x, \mu)\}}{1 - \text{Re tr} \{1 - U_0(x, \mu)^{-1} U(x, \mu)\} / f_R(\delta)} & \text{if } \text{Re tr} \{1 - U_0(x, \mu)^{-1} U(x, \mu)\} < f_R(\delta), \\ +\infty & \text{otherwise.} \end{cases} \quad (5.2)$$

The idea is that, with this mass term, gauge-field configurations are restricted within a ball with a radius  $\delta$  encircling a reference field  $U_0(x, \mu)$ . The reference field may be a non-trivial one such as the (lattice transcription of) instanton configuration. With  $\delta$  that fulfills eq. (2.12), we can introduce the one-parameter family

$$U_t(x, \mu) = U_0(x, \mu)[U_0(x, \mu)^{-1} U(x, \mu)]^t \in G, \quad 0 \leq t \leq 1, \quad (5.3)$$

which interpolates between the reference configuration  $U_0$  and the configuration under consideration,  $U$ . In contrast to the case of the vacuum sector, however, the inequality (2.13) is not enough to ensure that the ball is contained in the space of admissible configurations specified by eq. (2.7). The corresponding inequality must refer to  $U_0$ . Also, there seems no fundamental criterion to choose a particular reference configuration  $U_0$  within a topological sector. From these reasons, we do not pursue this generalization any further in this paper.

## 6. Conclusion

In this paper, we have presented a lattice framework with which one can study anomalous gauge theories with a Weyl fermion in an anomalous gauge representation. By introducing mass terms for all gauge bosons that impose smoothness on gauge degrees of freedom, we constructed a consistent fermion integration measure in the formulation of refs. [19, 20] for

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<sup>7</sup>See ref. [66] for a study of the non-linear sigma model on the lattice.

the vacuum sector of the configuration space of gauge fields. We argued that introduction of such (bare) mass terms is physically natural. Also, in view of the no go theorem in appendix A, we have to place a certain non gauge invariant restriction on lattice gauge-field configurations. An interesting question one can study with the present lattice framework is an upper bound on the UV cutoff in low-energy effective theories with anomalous fermion content that is suggested from a perturbative analysis [14].

In this paper, we have considered four-dimensional anomalous gauge theories. The present framework, when applied to two dimensions, can provide a lattice definition of the (off-critical) WZW model in two dimensions [67]. We hope to study this prospect in the near future.

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## A. A no go theorem for a Weyl fermion in an anomalous representation

The following no go theorem states that, under some assumptions, it is impossible to construct a physically sensible lattice formulation of a Weyl fermion for a wide class of anomalous gauge theories (that includes all four-dimensional non-abelian theories). The spacetime dimension is set to be  $2n$ .

**Theorem.** *Suppose that the compact gauge group  $G$  is semi-simple and  $\pi_1(G) = 0$  and  $\pi_{2n+1}(G) = \mathbb{Z}$ . Then if a Weyl fermion belongs to an anomalous representation  $R$  for which the leading anomaly coefficient  $A_{n+1}(A)$ , defined by*

$$\text{tr}\{R(F)^{n+1}\} = A_{n+1}(R) \text{tr}\{F^{n+1}\} + (\text{factorized traces}), \quad (\text{A.1})$$

where  $F$  is the field strength two-form in the fundamental representation, is non-vanishing  $A_{n+1}(R) \neq 0$ , the following four requirements are incompatible to each other:

1. *The lattice Weyl determinant reproduces the gauge anomaly in the classical continuum limit.*
2. *Only gauge invariant restrictions are placed on link variables.*
3. *The modulus of the lattice Weyl determinant is gauge invariant.*
4. *The lattice Weyl determinant is a (at least)  $C^2$ -class function of link variables.*

(Proof) We introduce a one-parameter family of lattice gauge transformations  $\Lambda_t(x) \in G$  ( $0 \leq t \leq 1$ ) such that  $\Lambda_0(x) = \Lambda_1(x) = 1$ . This one-parameter family  $\Lambda_t$  is a loop in the

space of lattice gauge transformations  $\mathfrak{G}$ . From this, we define a one-parameter family of pure-gauge link variables

$$U_t(x, \mu) = \Lambda_t(x)\Lambda_t(x + a\hat{\mu})^{-1}. \tag{A.2}$$

We then write the lattice Weyl determinant as

$$\langle 1 \rangle_{\mathbb{F}} [U] = r[U]e^{i\vartheta[U]} \tag{A.3}$$

and define the winding number of the complex phase around the loop

$$w = \frac{1}{2\pi} \int_0^1 dt \frac{\partial \vartheta[U_t]}{\partial t}. \tag{A.4}$$

Because of prerequisites in the theorem and the assumption 1, we can repeat an argument in the continuum [68]. It follows that (if the lattice is fine enough) we can choose the one-parameter family  $\Lambda_t(x)$  such that the winding number  $w$  is non-trivial,  $w \neq 0$ .

On the other hand, the space of lattice gauge transformations  $\mathfrak{G}$  is topologically trivial and any loop  $\Lambda_t$  is contractible ( $\pi_1(\mathfrak{G}) = 0$ ). Therefore, there exists a two-parameter family of lattice gauge transformations  $\Lambda_{t,s}(x)$  ( $0 \leq s \leq 1$ ) such that

$$\Lambda_{t,s=0}(x) = 1, \quad \Lambda_{t,s=1}(x) = \Lambda_t(x). \tag{A.5}$$

The corresponding two-parameter family of link variables

$$U_{t,s}(x, \mu) = \Lambda_{t,s}(x)\Lambda_{t,s}(x + a\hat{\mu})^{-1} \tag{A.6}$$

can then be regarded as a two-disk  $D$  in the space of gauge fields. Configurations belonging to this two-disk  $D$  are allowed configurations in a lattice formulation under consideration, because of the assumption 2. Note also that the one-parameter family (A.2) is the boundary of the disk  $D$ ,  $\partial D$ .

Now, from the assumptions 3, the modulus  $r$  is constant over the two-disk  $D$  (eq. (A.6)). Combined with the assumption 4, this fact implies that the one-form

$$a(t, s) = d\vartheta(t, s) \tag{A.7}$$

is a (at least)  $C^1$ -class function on  $D$ . However, then, from the Stokes theorem

$$w = \frac{1}{2\pi} \oint_{\partial D} a = \frac{1}{2\pi} \int_D da = 0 \tag{A.8}$$

(because  $da = dd\vartheta = 0$ ) and this is in contradiction with the above assertion that  $w \neq 0$ .  $\square$

The underlying physics for the above no go theorem is the following. In the continuum, an element of the gauge transformation  $\Lambda(x)$  must be a smooth function of the coordinate  $x$ . In lattice gauge theory, on the other hand,  $\Lambda(x)$  can take an arbitrary value in  $G$  at each site  $x$  and the field  $\Lambda(x)$  can be *arbitrarily random*. Roughly speaking, the space of lattice gauge fields is much larger than the space of continuum gauge fields, by the amount of random gauge degrees of freedom. In fact,  $\pi_1(\mathfrak{G}) = \pi_{2n+1}(G) \neq 0$  in the continuum and the two-parameter family (A.6) has no continuum analogue (i.e., the configurations are very

random at the cutoff scale). Any *gauge invariant* restriction, like the admissibility (2.7), cannot remove such very random configurations due to gauge degrees of freedom. The gauge degrees of freedom, even if they are very random, are harmless in usual gauge invariant lattice theories such as lattice QCD. In our present case of a Weyl fermion in an anomalous representation, however, gauge symmetry is broken and those random gauge degrees of freedom cause trouble which we do not encounter in the continuum.

The theorem indicates that we have only two physically sensible options. (It appears that the requirement 4 cannot be sacrificed because it ensures validity of the Schwinger-Dyson equations.) First is to abandon the requirement 3 and we allow the real part of the effective action to be gauge *variant*. This would indeed be the case if one uses the Wilson-Dirac operator. However, with the Wilson-Dirac operator, clear separation of left and right chiralities is impossible and we would go back to old controversy on Weyl nature of the lattice fermion. A manifest gauge invariance of the modulus of the lattice Weyl determinant is one of main achievements in the recent developments on lattice chiral gauge theories.

The second option is to abandon the requirement 2 and place some restriction on the link variables that suppresses random gauge degrees of freedom. This corresponds to the choice made in this paper; the smooth condition (2.11) is not gauge invariant and thus evades the theorem. One may then ask whether the condition (2.11) is enough for a physically sensible formulation. In the main text, we showed that this is indeed the case by explicitly constructing a smooth and local fermion integration measure of a Weyl fermion in the vacuum sector.

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